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Technical Notes

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Analysis of a Three-Dimensional Thermally Asymmetric Parallelepiped (Applicable to Fins)

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Nomenclature

Bi = Biot number, hl/k
h = heat transfer coefficient, W/m^2 °C
k = thermal conductivity, W/m °C
L = nondimensional parallelepided length (base to tip), L'/l
L' = parallelepiped length (base to tip), m
l = one-half parallelepiped height at the base, m
Q = heat loss from a parallelepiped, W
T = temperature, °C
T_w = base temperature, °C
T_∞ = ambient temperature, °C
w = nondimensional one-half parallelepiped width, w'/l
w' = one-half parallelepiped width, m
x = nondimensional length directional variable, x'/l
x' = length directional variable, m
y = nondimensional height directional variable, y'/l
y' = height directional variable, m
z = nondimensional width directional variable, z'/l
z' = width directional variable, m
θ = nondimensional temperature, $(T - T_\infty/(T_w - T_\infty)$
θ_0 = adjusted temperature, $(T_w - T_\infty)$
λ_n = eigenvalues; $n = 1, 2, 3, \ldots$
μ_m = eigenvalues; $m = 1, 2, 3, \ldots$
ρ_{nm} = eigenvalues, $\sqrt{(\lambda_n^2 + \mu_m^2)}$

Subscripts

1 = upper surface
2 = lower surface
3 = left surface
4 = right surface
5 = tip

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I. Introduction

THE classical shape of a parallelepiped has been considered in almost all advanced conduction heat transfer texts. However, the existing solutions are for special cases, for example, $T = T_w$ on one surface and $T = 0$ on all others. The more general problem, the case of convection from five of the six surfaces, would have particular application to fins. Hence, because the primary application of this shape would appear to be fins, this discussion will be directed at fins even though the presented solution will be applicable in general.

Fins are widely used to enhance the rate of heat transfer to a surrounding fluid in various industrial applications such as the cooling of combustion engines, electronic equipment, and various kinds of heat exchangers. As a result, a great deal of attention has been directed to fin problems. Various shapes of fins have been studied, including rectangular fins,[1–3] rectangular profile circular fins,[4] triangular fins,[5,6] and trapezoidal fins.[7,8] Finally in Refs. 9 and 10, there is inclusion of consideration of annular fins. Usually most of these studies assume that the heat transfer coefficients for all surfaces of the fin are the same constant, and most of them are analyzed using a one- or two-dimensional approach. No literature seems to be available that presents an analysis for a three-dimensional rectangular parallelepiped geometry, applicable to a fin, with unequal heat transfer coefficients on all of the surfaces.

This Note presents a three-dimensional analysis of a thermally asymmetric rectangular geometry. In this study Biot number Bi_1 is equal to or larger than Biot number Bi_2; Biot number Bi_3 is equal to or larger than Biot number Bi_4, and Biot number Bi_5 has various values. The nondimensional heat losses are investigated as a function of L, w, and Biot numbers Bi_5 and Bi_2/Bi_1 using the separation of variables method in three dimensions. Some comparisons of θ for thermally asymmetric conditions, and that for thermally symmetric conditions are made. Further, for arbitrary values of the L and the Biot numbers, the relation between the L and Biot numbers Bi_2/Bi_1 for equal amounts of heat loss are shown. Finally, the relation between the w and Biot numbers Bi_2/Bi_1 for equal amounts of heat loss is presented. For simplicity, the root temperature and the thermal conductivity of the fin material are assumed to be constant under steady-state operation.

II. Three-Dimensional Analysis

The geometry of a three-dimensional rectangular fin with all surfaces possessing different heat transfer coefficient is shown in Fig. 1. The normalized form of the three-dimensional governing differential equation for steady-state conditions is

$$\frac{\partial^2\theta}{\partial x^2} + \frac{\partial^2\theta}{\partial y^2} + \frac{\partial^2\theta}{\partial z^2} = 0 \tag{1}$$

The six boundary conditions, which are required to solve Eq. (1) are

$$\theta = 1 \qquad \text{at} \qquad x = 0 \tag{2}$$

$$\frac{\partial\theta}{\partial x} + Bi_5 \cdot \theta = 0 \qquad \text{at} \qquad x = L \tag{3}$$

$$\frac{\partial\theta}{\partial y} + Bi_1 \cdot \theta = 0 \qquad \text{at} \qquad y = 1 \tag{4}$$

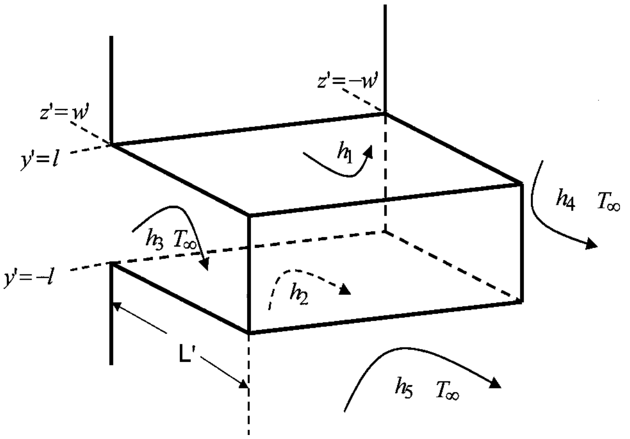

Fig. 1 Geometry of a three-dimensional thermally asymmetric rectangular fin.

$$\frac{\partial\theta}{\partial y} - Bi_2 \cdot \theta = 0 \qquad \text{at} \qquad y = -1 \tag{5}$$

$$\frac{\partial\theta}{\partial z} + Bi_3 \cdot \theta = 0 \qquad \text{at} \qquad z = w \tag{6}$$

$$\frac{\partial\theta}{\partial z} - Bi_4 \cdot \theta = 0 \qquad \text{at} \qquad z = -w \tag{7}$$

Please note that the uneven convection conditions (the five different h values) are handled through the Biot numbers and that the geometry presented as Fig. 1 dictates the signs in Eqs. (5) and (7). The form of the solution for the nondimensional temperature distribution $\theta(x, y, z)$ within the fin is

$$\theta(x, y, z) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} N_{nm} \cdot f_1(x) \cdot f_2(y) \cdot f_3(z) \tag{8}$$

where

$$N_{nm} = \frac{4 \sin\lambda_n \cdot \sin(u_m \cdot w)}{f_n \cdot g_m} \tag{9}$$

$$f_1(x) = \cosh(\rho_{nm} \cdot x) - C_{nm} \cdot \sinh(\rho_{nm} \cdot x) \tag{10}$$

$$C_{nm} = \frac{\rho_{nm} \cdot \tanh(\rho_{nm} \cdot L) + Bi_5}{\rho_{nm} + Bi_5 \cdot \tanh(\rho_{nm} \cdot L)} \tag{11}$$

$$\rho_{nm} = \sqrt{\left(\lambda_n^2 + \mu_m^2\right)} \tag{12}$$

$$f_2(y) = \cos(\lambda_n \cdot y) + A_n \cdot \sin(\lambda_n \cdot y) \tag{13}$$

$$f_3(z) = \cos(\mu_m \cdot z) + B_m \cdot \sin(\mu_m \cdot z) \tag{14}$$

In the limit this solution reduces to those special cases presented in Refs. 10–12. Equation (15) was derived using the boundary condition of Eq. (4), whereas Eq. (16) comes from Eq. (6). Thus,

$$A_n = \frac{\lambda_n \cdot \tan\lambda_n - Bi_1}{\lambda_n + Bi_1 \cdot \tan\lambda_n} \tag{15}$$

$$B_m = \frac{\mu_m \cdot \tan(\mu_m \cdot w) - Bi_3}{\mu_m + Bi_3 \cdot \tan(\mu_m \cdot w)} \tag{16}$$

Finally, f_n and g_m appearing in Eq. (9) can be written as

$$f_n = \lambda_n + \tfrac{1}{2}\sin(2\lambda_n) + A_n^2 \cdot \left\{\lambda_n - \tfrac{1}{2}\sin(2\lambda_n)\right\} \tag{17}$$

$$g_m = \mu_m \cdot w + \tfrac{1}{2}\sin(\mu_m \cdot w) + B_m^2 \cdot \left\{\mu_m \cdot w - \tfrac{1}{2}\sin(\mu_m \cdot w)\right\} \tag{18}$$

The eigenvalues λ_n obtained from Eq. (19) [which is derived from Eq. (15)] are based on the boundary condition of Eq. (5) and relates the top and bottom surfaces conditions:

$$\frac{\lambda_n \cdot \sin\lambda_n - Bi_1 \cdot \cos\lambda_n}{\lambda_n \cdot \cos\lambda_n + Bi_1 \cdot \sin\lambda_n} = \frac{Bi_2 \cdot \cos\lambda_n - \lambda_n \cdot \sin\lambda_n}{\lambda_n \cdot \cos\lambda_n + Bi_2 \cdot \sin\lambda_n} \tag{19}$$

Similarly, the eigenvalues μ_m obtained from Eq. (20) [which is derived from Eq. (16)] are based on the boundary condition of Eq. (7) and relates the right- and left-side surfaces conditions:

$$\frac{\mu_m \cdot \sin(\mu_m \cdot w) - Bi_3 \cdot \cos(\mu_m \cdot w)}{\mu_m \cdot \cos(\mu_m \cdot w) + Bi_3 \cdot \sin(\mu_m \cdot w)} = \frac{Bi_4 \cdot \cos(\mu_m \cdot w) - \mu_m \cdot \sin(\mu_m \cdot w)}{\mu_m \cdot \cos(\mu_m \cdot w) + Bi_4 \cdot \sin(\mu_m \cdot w)} \tag{20}$$

With the temperature profile, the heat loss from the fin is obtained by evaluating Fourier's law at the root. Thus,

$$\frac{Q}{kl\theta_0} = 4\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} N_{nm} \cdot \rho_{nm} \cdot C_{nm} \cdot \frac{\sin\lambda_n}{\lambda_n} \cdot \frac{\sin(\mu_m \cdot w)}{\mu_m} \tag{21}$$

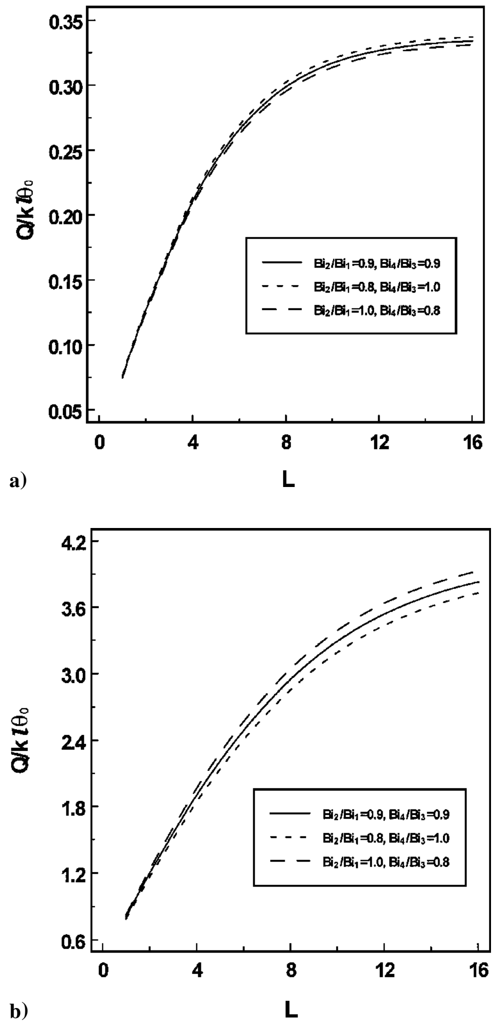

Fig. 2 Nondimensional heat loss vs L in the case of $Bi_1 = Bi_3 = Bi_5 = 0.01$ for a) $w = 0.5$ and b) $w = 10$.

III. Results and Discussion

For Figs. 2–6, the selected values of the parameters are characteristic of the use in fins. However, recall that the solution presented is true in general.

Figure 2 presents the nondimensional heat loss from a thermally asymmetric rectangular fin vs L for $Bi_1 = Bi_3 = Bi_5 = 0.01$ and three cases of the ratios of Biot number ratios Bi_2/Bi_1 and Bi_4/Bi_3. These values were arbitrarily chosen, and the average of these Biot numbers ratios for these three cases are equal. Note that in Fig. 2a the heat loss for $Bi_2/Bi_1 = 0.8$, $Bi_4/Bi_3 = 1.0$ is the highest, whereas that for $Bi_2/Bi_1 = 1.0$, $Bi_4/Bi_3 = 0.8$ is the lowest, and the difference between these values becomes larger as L increases. Figure 2b indicates that the heat losses in case of $w = 10$ are larger than for $w = 0.5$, and the trends for heat loss are somewhat similar for both values of w. However, the heat loss for $w = 10$ and $Bi_2/Bi_1 = 0.8$, $Bi_4/Bi_3 = 1.0$ is the lowest, whereas that for $Bi_2/Bi_1 = 1.0$, $Bi_4/Bi_3 = 0.8$ is the highest, and this phenomenon is not unexpected because the fin has wide top and bottom surfaces.

Figure 3 shows the same sort of relationship as presented in Figs. 2, but as a function of w in the case of $Bi_1 = Bi_3 = Bi_5 = 0.01$ and $L = 20$. The heat losses increase linearly as w increases, and the value for $Bi_2/Bi_1 = 0.8$, $Bi_4/Bi_3 = 1.0$ is the lowest, whereas that for $Bi_2/Bi_1 = 1.0$, $Bi_4/Bi_3 = 0.8$ is the highest in the range of $w > 1$.

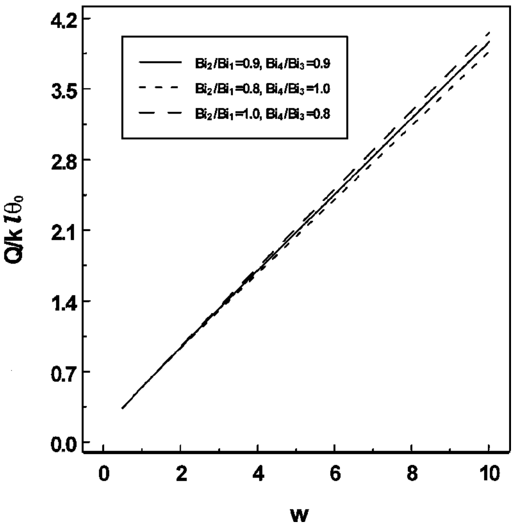

Fig. 3 Nondimensional heat loss vs w for $L = 20$ and $Bi_1 = Bi_3 = Bi_5 = 0.01$.

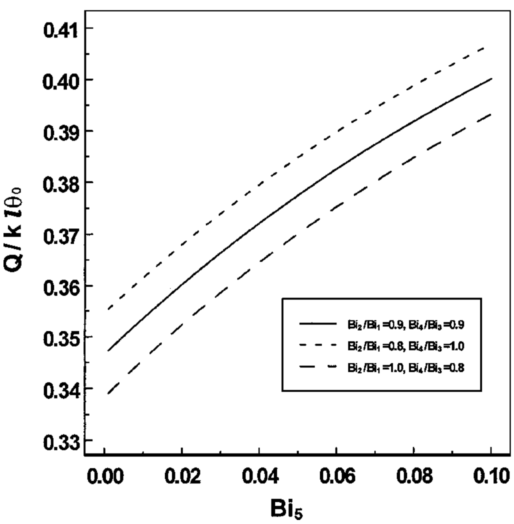

Fig. 4 Nondimensional heat loss vs Bi_5 for $w = 0.5$, $L = 5$, $Bi_1 = 0.01$, and $Bi_3 = 0.02$.

Table 1 Ratio of temperatures at various position for thermally asymmetric case compared to symmetric case[a]

	$z = 0.5$		$z = 0$		$z = -0.5$	
x	$y = 1$	$y = -1$	$y = 1$	$y = -1$	$y = 1$	$y = -1$
0.1	98.92	100.76	99.08	100.92	99.24	101.08
1	98.63	100.49	99.08	100.95	99.54	101.41
2	98.62	100.49	99.10	100.97	99.58	101.46
3	98.64	100.50	99.12	100.99	99.60	101.48
4	98.65	100.51	99.13	101.01	99.61	101.50
5	98.66	100.53	99.14	101.01	99.62	101.51

[a]Here θ for $Bi_1 = Bi_3 = 0.11$, $Bi_2 = Bi_4 = 0.09$, and $Bi_5 = 0.1/\theta$ for

$$Bi(\text{sym}) = \sum \frac{Bi(\text{asym})}{5}$$

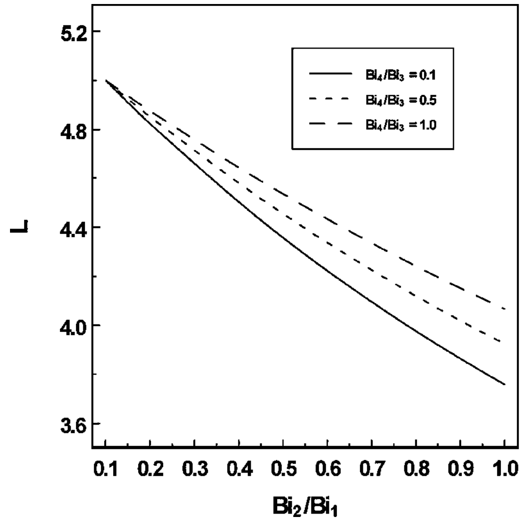

Fig. 5 Relation between L and Biot number ratio Bi_2/Bi_1 for equal amounts of heat loss based on $Bi_1 = Bi_3 = Bi_5 = 0.01$ and $w = 0.5$.

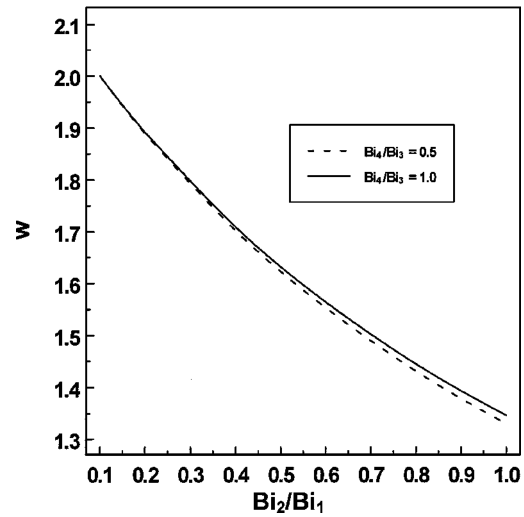

Fig. 6 Relation between w and Biot number ratio Bi_2/Bi_1 for equal amounts of heat loss based on $Bi_1 = Bi_3 = Bi_5 = 0.01$ and $L = 5$.

Figure 4 presents the nondimensional heat loss vs Biot number B_5 for $Bi_1 = 0.01$, $Bi_3 = 0.02$, $w = 0.5$, and $L = 5$. Notice that when Biot number Bi_3 is larger than Biot number Bi_1, heat loss for $Bi_2/Bi_1 = 0.8$, $Bi_4/Bi_3 = 1.0$ is the highest for the given range of Biot number Bi_5. Furthermore, the heat losses for all three cases increase continuously as Biot number Bi_5 increases.

Table 1 lists the ratio of temperatures at various positions in and on the fin surfaces for a thermally asymmetric case and the thermally symmetric case when $L=5$ and $w=0.5$. The value of the symmetric case Biot number is the average of the Biot numbers in the asymmetric case. As expected, the temperature ratio along the fin length is the lowest at $y=1$, $z=0.5$ and is the highest at $y=-1$, $z=-0.5$.

The relationships between L and Biot number ratio Bi_2/Bi_1 for equal amounts of heat loss based on $Bi_1=Bi_3=Bi_5=0.01$ and $w=0.5$ are shown in Fig. 5. The trends of L with Biot number ratio Bi_2/Bi_1 have negative slopes for the three cases of Biot number ratio Bi_4/Bi_3 illustrated. Furthermore, Fig. 5 shows that the slope becomes more negative as the Biot number ratio of Bi_4/Bi_3 decreases. Figure 6 presents the relationships between the w and Biot number ratio Bi_2/Bi_1 for equal amounts of heat loss based on $Bi_1=Bi_3=Bi_5=0.01$ and $L=5$. The value of w decreases as Biot number ratio Bi_2/Bi_1 increases. When Fig. 6 is compared to Fig. 5, the slope of Fig. 6 becomes more negative as Biot number ratio Bi_4/Bi_3 decreases but the difference between both values, that is, $Bi_4/Bi_3=5$ and 1, are relatively small.

IV. Conclusions

The following conclusions can be drawn from these results.

1) Heat loss increases nonlinearly as fin length increases, whereas it increases linearly as fin width increases.

2) When the average of the Biot number ratios Bi_2/Bi_1 and Bi_4/Bi_3 are the same, there is more heat loss in the case of asymmetric top to bottom and symmetric right to left than vice versa when w is small, and this relationship reverses as w increases. The crossover point appears to be approximately $w=1$, which represents the dimensionally invariant point $w'=l$.

3) The fin length and width must decrease as Biot number ratio Bi_2/Bi_1 increases to produce equal amount of heat loss and the slope increases as Biot number ratio Bi_4/Bi_3 decreases.

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Demythicizing the Gibbs Potential for Determination of Chemical Equilibrium Conditions

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Introduction

It is well known that the chemical equilibrium of a multicomponent system is characterized by the minimization of thermodynamic functions such as internal energy U, enthalpy H, Helmholtz potential F, and Gibbs potential G

$$(\mathrm{d}U)_{S,V}=(\mathrm{d}H)_{S,p}=(\mathrm{d}F)_{T,V}=(\mathrm{d}G)_{T,p}=\sum_i^n \mu_i\,\mathrm{d}N_i=0 \qquad (1)$$

where μ_i is the molar chemical potential defined as

$$\mu_i=\left(\frac{\partial U}{\partial N_i}\right)_{S,V,N_{j\neq i}}=\left(\frac{\partial H}{\partial N_i}\right)_{S,p,N_{j\neq i}}=\left(\frac{\partial F}{\partial N_i}\right)_{T,V,N_{j\neq i}}=\left(\frac{\partial G}{\partial N_i}\right)_{T,p,N_{j\neq i}} \qquad (2)$$

The same characterization can be achieved by resorting to the maximization of the entropy and the relative entropic potentials.[1] The equivalence of the different formulations should appear clear from Eq. (1). Nevertheless, some misunderstandings appear in the literature because the thermodynamic constraints [subscripts in Eq. (1)] are most often considered a consequence of the physical transformation followed by the system toward the finale state of equilibrium, rather than a virtual mathematical procedure. In this regard, the chemist favors the minimization of the Gibbs potential whereas the physicist is more used to minimizing the Helmholtz potential, without fully perceiving that, within a specific equilibrium problem, both criteria yield the same equilibrium conditions. On the other hand, thermodynamics textbooks also show that the gas-phase equilibrium constants K_p, $K_c, \ldots$, are connected by simple relations such as

$$K_p=\exp(-\Delta\mu^\circ/RT)=K_c\cdot(RT)^{\Delta\nu}=\cdots \qquad (3)$$

In Eq. (3), K_p and K_c are the equilibrium constants in terms of partial pressures and concentrations, $\Delta\mu^\circ$ is the reaction standard free energy, and $\Delta\nu$ is the difference between the sums of the stoichiometric coefficients of the products and the reactants, respectively. Researchers, therefore, use either K_p or K_c, regardless of the transformation that the system undergoes; as a matter of fact, they use the minimization of any one of the thermodynamic functions appearing in Eq. (1), in flagrant contradiction with the textbooks that stress the importance of using the different constants according to the particular transformation that the system experiences. In particular,

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